# Fuzzy Frequency Response for Complex Dynamic Systems

Carlos Cesar Teixeira Ferreira and Ginalber Luiz de Oliveira Serra

***Abstract*— Fuzzy Frequency Response: Definition and Analysis for Complex Dynamic Systems is proposed in this paper. In terms of transfer function, the complex dynamic system is partitioned into several linear sub-models and it is organized into Takagi-Sugeno (TS) fuzzy structure. The main contribution of this approach is demonstrated, from the proposal of a *Theorem*, that fuzzy frequency response is a boundary in the magnitude and phase Bode plots. Low and high frequency analysis of fuzzy dynamic model is obtained by varying the frequency $\omega$ from zero to infinity.**

## I. Introduction

The main task of the control theory is the analysis and design for complex dynamic systems. In the analysis, the characteristics or dynamic behaviour of the control system are determined. In the design, the controllers are obtained to attend the desired characteristics of the control system from certain performance criteria. Generally, these criteria may involve disturbance rejection, steady-state errors, transient response characteristics and sensitivity to parameter changes in the plant [1], [2], [3]. Since the real environment may vary with time or its operating conditions may change with load and disturbances, the control system must be able to withstand these variations. The particular property that a control system must possess in order to operate properly in this real environment is called robustness. The robust analysis and control techniques are conveniently examined in the frequency domain. The frequency response methods were developed during the period $1930 - 1940$ by Harry Nyquist $(1889-1976)$ [5], Hendrik Bode $(1905-1982)$ [6], Nathaniel B. Nichols $(1914 - 1997)$ [4] and many others. Since, frequency response methods are among the most useful techniques and available to analyse and synthesise the controllers. In [7], the U.S. Navy obtains frequency responses for aircraft by applying sinusoidal inputs to the autopilots and measuring the resulting position of the aircraft while the aircraft is in flight. In [8], four current controllers for selective harmonic compensation in parallel Active Power Filters (APFs) have been compared analytically in terms of frequency response characteristics and maximum operational frequency. A complex dynamic system presents uncertainty and/or nonlinearity in its dynamic behaviour. This paper proposes the definition of Fuzzy Frequency Response (FFR) and its application for analysis of complex dynamic systems. The complex dynamic system is partitioned into several linear sub-models and it is organized into Takagi-Sugeno (TS) fuzzy structure. The main contribution of this approach is demonstrated, from the proposal of a *Theorem*, that fuzzy frequency response is a boundary in the magnitude and phase Bode plots. Low and high frequency analysis of fuzzy dynamic model is obtained by varying the frequency $\omega$ from zero to infinity.

The paper is organized as follows: an overview of Takagi-Sugeno Fuzzy Dynamic Model is first given in Section II. The definition of fuzzy frequency response is addressed in Section III. Fuzzy frequency response at low and high frequencies is analyzed in Section IV. Section V presents the computational results for fuzzy frequency response of two complex systems with uncertain and nonlinear dynamic behaviour, respectively. Final remarks are given in Section VI.

## II. Takagi-Sugeno Fuzzy Dynamic Model

The inference system TS, originally proposed in [9], presents in the consequent a dynamic functional expression of the linguistic variables of the antecedent. The $i\Big|^{[i=1,2,...,l]}$-th rule, where $l$ is the rules numbers, is given by

$$Rule^{(i)}:$$
$$IF\ \tilde{x}_1 \text{ is } F^i_{\{1,2,...,p_{\tilde{x}_1}\}|\tilde{x}_1}\ AND \ldots AND\ \tilde{x}_n \text{ is } F^i_{\{1,2,...,p_{\tilde{x}_n}\}|\tilde{x}_n}$$
$$\text{THEN } y_i = f_i(\tilde{\mathbf{x}}) \quad (1)$$

where the total number of rules is $l = p_{\tilde{x}_1} \times \ldots \times p_{\tilde{x}_n}$. The vector $\tilde{\mathbf{x}} = [\tilde{x}_1, \ldots, \tilde{x}_n]^T \in \Re^n$ containing the linguistics variables of antecedent, where $T$ represents the operator for transpose matrix. Each linguistic variable has its own discourse universe $\mathcal{U}_{\tilde{x}_1}, \ldots, \mathcal{U}_{\tilde{x}_n}$, partitioned by fuzzy sets representing its linguistics terms, respectively. In $i$-th rule, the variable $\tilde{x}_{\{1,2,...,n\}}$ belongs to the fuzzy set $F^i_{\{\tilde{x}_1,...,\tilde{x}_n\}}$ with a membership degree $\mu^i_{F_{\{\tilde{x}_1,...,\tilde{x}_n\}}}$ defined by a membership function $\mu^i_{\{\tilde{x}_1,...,\tilde{x}_n\}} : \Re \rightarrow [0,1]$, with $\mu^i_{F_{\{\tilde{x}_1,...,\tilde{x}_n\}}} \in \{\mu^i_{F_{1|\{\tilde{x}_1,...,\tilde{x}_n\}}}, \mu^i_{F_{2|\{\tilde{x}_1,...,\tilde{x}_n\}}}, \ldots, \mu^i_{F_{p|\{\tilde{x}_1,...,\tilde{x}_n\}}}\}$, where $p_{\{\tilde{x}_1,...,\tilde{x}_n\}}$ is the partitions number of the discourse universe associated with the linguistic variable $\tilde{x}_1, \ldots, \tilde{x}_n$. The output of the TS fuzzy dynamic model is a convex combination of the dynamic functional expressions of consequent $f_i(\tilde{\mathbf{x}})$, without loss of generality for the bidimensional case, as illustrated in Fig. 1, given by Eq. (2).

$$y(\tilde{\mathbf{x}}, \gamma) = \sum_{i=1}^{l} \gamma_i(\tilde{\mathbf{x}}) f_i(\tilde{\mathbf{x}}) \quad (2)$$

where $\gamma$ is the scheduling variable of the TS fuzzy dynamic model. The scheduling variable, well known as normalized activation degree is given by:

Carlos Cesar Teixeira Ferreira is with the Department of Electroelectronic, Federal Institute of Education, Science and Technology (IFMA), Campus Monte Castelo, 04, Getúlio Vargas Av., Monte Castelo, São Luís-MA, Brazil (phone: +55 98 32189088; email: ccteixeira@ifma.edu.br).

Ginalber Luiz de Oliveira Serra is with the Department of Electroelectronic, Federal Institute of Education, Science and Technology (IFMA), Campus Monte Castelo, 04, Getúlio Vargas Av., Monte Castelo, São Luís-MA, Brazil (phone: +55 98 32189088; email: ginalber@ifma.edu.br).

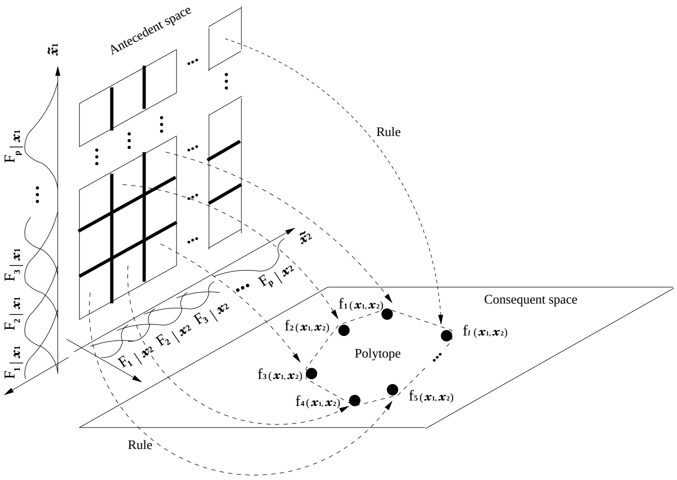


Fig. 1. Fuzzy dynamic model: A TS model can be regarded as a mapping from the antecedent space to the space of the consequent parameters one.

$$\gamma_i(\tilde{\mathbf{x}}) = \frac{h_i(\tilde{\mathbf{x}})}{\sum_{r=1}^{l} h_r(\tilde{\mathbf{x}})}. \tag{3}$$

This normalization implies

$$\sum_{k=1}^{l} \gamma_i(\tilde{\mathbf{x}}) = 1. \tag{4}$$

It can be observed that the TS fuzzy dynamic system, which represents any nonlinear dynamic model, may be considered as a class of systems where $\gamma_i(\tilde{\mathbf{x}})$ denotes a decomposition of linguistic variables $[\tilde{x}_1, \ldots, \tilde{x}_n]^T \in \Re^n$ for a polytopic geometric region in the consequent space from the functional expressions $f_i(\tilde{\mathbf{x}})$.

## III. Fuzzy Frequency Response (FFR): Definition

This section will present how a TS fuzzy model of a complex dynamic system responds to sinusoidal inputs, which in this paper is proposed as the definition of fuzzy frequency response. The response of a TS fuzzy model to a sinusoidal input of frequency $\omega_1$ in both amplitude and phase, is given by the transfer function evaluated at $s = j\omega_1$, as illustrated in Fig. 2.

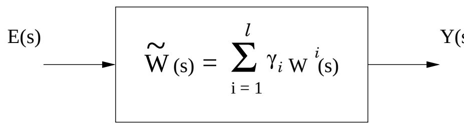


Fig. 2. TS fuzzy transfer function

For this TS fuzzy model,

$$Y(s) = \left[\sum_{i=1}^{l} \gamma_i W^i(s)\right] E(s). \tag{5}$$

Consider $\tilde{W}(j\omega) = \sum_{i=1}^{l} \gamma_i W^i(j\omega)$ a complex number for a given $\omega$, as

$$\tilde{W}(j\omega) = \sum_{i=1}^{l} \gamma_i W^i(j\omega) = \left|\sum_{i=1}^{l} \gamma_i W^i(j\omega)\right| e^{j\phi(\omega)} \tag{6}$$

or

$$\tilde{W}(j\omega) = \sum_{i=1}^{l} \gamma_i W^i(j\omega) = \left|\sum_{i=1}^{l} \gamma_i W^i(j\omega)\right| \angle \arctan \left[\sum_{i=1}^{l} \gamma_i W^i(j\omega)\right]. \tag{7}$$

Then, for the case that the input signal $e(t)$ is sinusoidal, that is,

$$e(t) = A \sin \omega_1 t. \tag{8}$$

The output signal $y_{ss}(t)$, in the steady state, is given by

$$y_{ss}(t) = A \left|\sum_{i=1}^{l} \gamma_i W^i(j\omega)\right| \sin\left[\omega_1 t + \phi(\omega_1)\right]. \tag{9}$$

As result of the fuzzy frequency response definition, it is proposed the following *Theorem*:

*Theorem 3.1:* Fuzzy frequency response is a region in the frequency domain, defined by the consequent sub-models and from the operating region of the antecedent space.

*Proof:*

Considering that $\tilde{\nu}$ is a linguistic variable related to the dynamic behaviour of the complex system, it can be represented by linguistic terms. Once known its discourse universe, as shown in Fig. 3, the activation degrees $h_i(\tilde{\nu})|^{i=1,2,\ldots,l}$ are given by:

$$h_i(\tilde{\nu}) = \mu^i_{F_{\tilde{\nu}^*_1}} \star \mu^i_{F_{\tilde{\nu}^*_2}} \star \ldots \star \mu^i_{F_{\tilde{\nu}^*_n}}, \tag{10}$$

where $\tilde{\nu}^*_{\{1,2,\ldots,n\}} \in \mathcal{U}_{\tilde{\nu}_{\{1,2,\ldots,n\}}}$, respectively, and $\star$ is a fuzzy logic operator. The normalized activation degrees $\gamma_i(\tilde{\nu})|^{i=1,2,\ldots,l}$, are also related to the dynamic behaviour as follow:

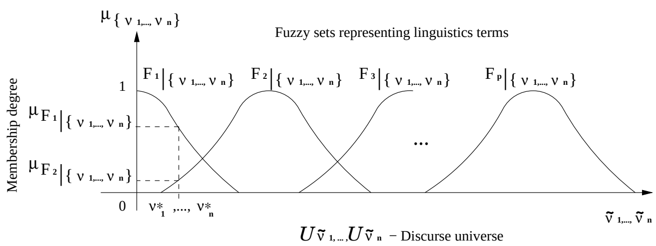


Fig. 3. Functional description of the linguistic variable $\tilde{\nu}$: linguistic terms, discourse universes and membership degrees.

$$\gamma_i(\tilde{\nu}) = \frac{h_i(\tilde{\nu})}{\sum_{r=1}^{l} h_r(\tilde{\nu})}. \tag{11}$$

This normalization implies

$$\sum_{k=1}^{l} \gamma_i(\tilde{\nu}) = 1. \tag{12}$$

Let $F(s)$ be a vectorial space with degree $l$ and $f^1(s), f^2(s), \ldots, f^l(s)$ transfer functions which belongs to this vectorial space. A transfer function $f(s) \in F(s)$ must be a linear convex combination of the vectors $f^1(s), f^2(s), \ldots, f^l(s)$:

$$f(s) \;=\; \xi_1 f^1(s) + \xi_2 f^2(s) + \ldots + \xi_l f^l(s), \tag{13}$$

where $\xi_{1,2,\ldots,l}$ are the coefficients of this linear convex combination. If the coefficients of the linear convex combination are normalized $\left(\sum_{i=1}^{l} \xi_i = 1\right)$, the vectorial space presents a decomposition of the transfer functions $\left[f^1(s),\ \ f^2(s), \ldots, f^l(s)\right]$ in a polytopic geometric shape of the vectorial space $F(s)$. The points of the polytopic geometric shape are defined by the transfer functions $\left[f^1(s), f^2(s), \ldots, f^l(s)\right]$. The TS fuzzy dynamic model attends this polytopic property. The sum of the normalized activation degrees is equal to 1, as shown in Eq. (4). To define the points of this fuzzy polytopic geometric shape, each rule of the TS fuzzy dynamic model must be singly activated. This condition is called boundary condition. In this way, the following results are obtained for the Fuzzy Frequency Response (FFR) of the TS fuzzy transfer function:

- *If only the rule* 1 *is activated, it has* $(\gamma_1 = 1, \gamma_2 = 0, \gamma_3 = 0, \ldots, \gamma_l = 0)$*:*

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|\sum_{i=1}^{l} \gamma_i(\tilde{\nu}) W^i(j\omega)\right| \angle \arctan\left[\sum_{i=1}^{l} \gamma_i(\tilde{\nu}) W^i(j\omega)\right], \tag{14}$$

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|1W^1(j\omega) + 0W^2(j\omega) + \ldots + 0W^l(j\omega)\right| \angle \arctan \left[1W^1(j\omega) + 0W^2(j\omega) + \ldots + 0W^l(j\omega)\right],$$

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|W^1(j\omega)\right| \angle \arctan\left[W^1(j\omega)\right]. \tag{15}$$

- *If only the rule* 2 *is activated, it has* $(\gamma_1 = 0, \gamma_2 = 1, \gamma_3 = 0, \ldots, \gamma_l = 0)$*:*

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|\sum_{i=1}^{l} \gamma_i(\tilde{\nu}) W^i(j\omega)\right| \angle \arctan\left[\sum_{i=1}^{l} \gamma_i(\tilde{\nu}) W^i(j\omega)\right], \tag{16}$$

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|0W^1(j\omega) + 1W^2(j\omega) + \ldots + 0W^l(j\omega)\right| \angle \arctan \left[0W^1(j\omega) + 1W^2(j\omega) + \ldots + 0W^l(j\omega)\right],$$

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|W^2(j\omega)\right| \angle \arctan\left[W^2(j\omega)\right]. \tag{17}$$

- *If only the rule* $l$ *is activated, it has* $(\gamma_1 = 0, \gamma_2 = 0, \gamma_3 = 0, \ldots, \gamma_l = 1)$*:*

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|\sum_{i=1}^{l} \gamma_i(\tilde{\nu}) W^i(j\omega)\right| \angle \arctan\left[\sum_{i=1}^{l} \gamma_i(\tilde{\nu}) W^i(j\omega)\right], \tag{18}$$

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|0W^1(j\omega) + 0W^2(j\omega) + \ldots + 1W^l(j\omega)\right| \angle \arctan \left[0W^1(j\omega) + 0W^2(j\omega) + \ldots + 1W^l(j\omega)\right],$$

$$\tilde{W}(j\omega,\tilde{\nu}) = \left|W^l(j\omega)\right| \angle \arctan\left[W^l(j\omega)\right], \tag{19}$$

where $W^1(j\omega), W^2(j\omega), \ldots, W^l(j\omega)$ are the linear sub-models of the complex dynamic system. Note that $\left|W^1(j\omega)\right| \angle \arctan\left[W^1(j\omega)\right]$ and $\left|W^l(j\omega)\right| \angle \arctan\left[W^l(j\omega)\right]$ define a boundary region. Under such circumstances the fuzzy frequency response for complex dynamic system converges to a boundary in the frequency domain. Figure 4 shows the fuzzy frequency response for the bidimensional case, without loss of generality.

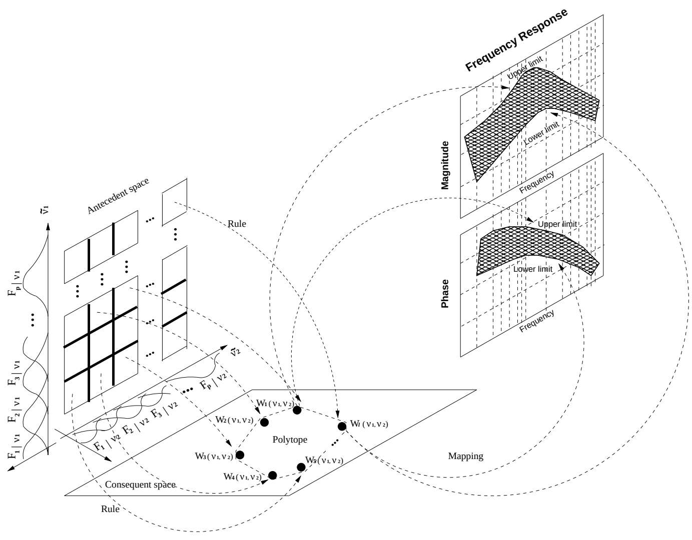


Fig. 4. Fuzzy frequency response: mapping from the consequent space to the region in the frequency domain.

■

## IV. Fuzzy Frequency Response (FFR): Analysis

In this section will be analyzed the behavior of the fuzzy frequency response at low and high frequencies. The idea is to study the magnitude and phase behavior of the TS fuzzy dynamic model, when $\omega$ varies from zero to infinity.

### A. Low Frequencies Analysis

The low frequencies analysis of the TS fuzzy dynamic model $\tilde{W}(s)$ can be obtained by

$$\lim_{\omega\to 0}\sum_{i=1}^{l}\gamma_i W^i(j\omega). \tag{20}$$

The magnitude and phase behaviour at low frequencies, is given by

$$\lim_{\omega\to 0}\left|\sum_{i=1}^{l}\gamma_i W^i(j\omega)\right| \angle \arctan\left[\sum_{i=1}^{l}\gamma_i W^i(j\omega)\right]. \tag{21}$$

### B. High Frequencies Analysis

Equivalently, the high frequencies analysis of the TS fuzzy dynamic model $\tilde{W}(s)$ can be obtained by

$$\lim_{\omega\to \infty}\sum_{i=1}^{l}\gamma_i W^i(j\omega). \tag{22}$$

The magnitude and phase behaviour at high frequencies, is given by

$$\lim_{\omega\to \infty}\left|\sum_{i=1}^{l}\gamma_i W^i(j\omega)\right| \angle \arctan\left[\sum_{i=1}^{l}\gamma_i W^i(j\omega)\right]. \tag{23}$$

## V. Computational Results

To illustrate the FFR: definition and analysis, as shown in section III and IV, consider two cases of complex system:

- Complex system with uncertain dynamic behaviour;
- Complex system with nonlinear dynamic behaviour.

### A. Uncertain Dynamic System

Consider the following uncertain dynamic system, given by

$$H(s,\nu) = \frac{Y(s,\nu)}{U(s)} = \\ = \frac{2-\nu}{[(\nu+1)s+1]\left[\left(\frac{\nu}{2}+0.1\right)s+1\right]} \tag{24}$$

where the scheduling variable is $\nu = [0,1]$, the gain of the uncertain dynamic system is $K_p = 2-\nu$, the upper time constant is $\tau = \nu + 1$ and the lower time constant is $\tau' = \frac{\nu}{2} + 0.1$. From the uncertain dynamic system in Eq. (24) and assuming the time varying scheduling variable in the range of $[0,1]$, it can obtain the TS fuzzy dynamic model in the following operating points:

**Sub-model 1 ($\nu = 0$):**

$$W^1(s,0) = \frac{2}{(s+1)(0.1s+1)} = \frac{2}{0.1s^2+1.1s+1}. \tag{25}$$

**Sub-model 2 ($\nu = 0.5$):**

$$W^2(s,0.5) = \frac{1.5}{(1.5s+1)(0.35s+1)} = \frac{1.5}{0.525s^2+1.85s+1}. \tag{26}$$

**Sub-model 3 ($\nu = 1$):**

$$W^3(s,1) = \frac{1}{(2s+1)(0.6s+1)} = \frac{1}{1.2s^2+2.6s+1}. \tag{27}$$

The TS fuzzy dynamic model rules base results

$$\begin{array}{llllll} Rule^{(1)}: & \text{IF} & \nu \text{ is} & 0 & \text{THEN} & W^1(s,0) \\ Rule^{(2)}: & \text{IF} & \nu \text{ is} & 0.5 & \text{THEN} & W^2(s,0.5) \\ Rule^{(3)}: & \text{IF} & \nu \text{ is} & 1 & \text{THEN} & W^3(s,1), \end{array} \tag{28}$$

From Eq. (7) the TS fuzzy dynamic model of the uncertain dynamic system, Eq. (28), can be represented by

$$\tilde{W}(j\omega,\tilde{\nu}) = \\ = \left|\sum_{i=1}^{3}\gamma_i(\tilde{\nu})W^i(j\omega)\right| \angle \arctan\left[\sum_{i=1}^{3}\gamma_i(\tilde{\nu})W^i(j\omega)\right] \tag{29}$$

So,

$$\tilde{W}(j\omega,\tilde{\nu}) = \\ = \left|\gamma_1\frac{2}{0.1s^2+1.1s+1} + \gamma_2\frac{1.5}{0.525s^2+1.85s+1} + \right. \\ \left. +\gamma_3\frac{1}{1.2s^2+2.6s+1}\right| \angle \arctan\left[\gamma_1\frac{2}{0.1s^2+1.1s+1} + \right. \\ \left. +\gamma_2\frac{1.5}{0.525s^2+1.85s+1} + \gamma_3\frac{1}{1.2s^2+2.6s+1}\right] \tag{30}$$

$$\tilde{W}(j\omega,\tilde{\nu}) = \\ \left|2\gamma_1\frac{0.6(j\omega)^4+3.6(j\omega)^3+6.5(j\omega)^2+4.5(j\omega)+1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \right. \\ 1.5\gamma_2\frac{0.1(j\omega)^4+1.6(j\omega)^3+4.2(j\omega)^2+3.7(j\omega)+1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \\ \left.\gamma_3\frac{0.1(j\omega)^4+0.8(j\omega)^3+2.7(j\omega)^2+3(j\omega)+1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}}\right| \angle \arctan \\ \left[2\gamma_1\frac{0.6(j\omega)^4+3.6(j\omega)^3+6.5(j\omega)^2+4.5(j\omega)+1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \right. \\ 1.5\gamma_2\frac{0.1(j\omega)^4+1.6(j\omega)^3+4.2(j\omega)^2+3.7(j\omega)+1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} +$$

$$\gamma_3 \frac{0.1(j\omega)^4 + 0.8(j\omega)^3 + 2.7(j\omega)^2 + 3(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} \Bigg] \quad (31)$$

where

$$Den_{[\tilde{W}(j\omega,\tilde{\nu})]} = 0.1(j\omega)^6 + 1.1(j\omega)^5 + 5.2(j\omega)^4 + $$
$$+ 11.2(j\omega)^3 + 11.5(j\omega)^2 + 5.6(j\omega) + 1 \quad (32)$$

*1) Low Frequencies Analysis:* From the TS fuzzy dynamic model, Eq. (29), and applying the concepts seen in the Subsection IV-A, the steady-state response for sinusoidal input at low frequencies for the uncertain dynamic system can be obtained as follow:

$$\lim_{\omega \to 0} \tilde{W}(j\omega,\tilde{\nu}) = \Bigg| 2\gamma_1 \frac{0.6(j\omega)^4 + 3.6(j\omega)^3 + 6.5(j\omega)^2 + 4.5(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + 1.5\gamma_2 \frac{0.1(j\omega)^4 + 1.6(j\omega)^3 + 4.2(j\omega)^2 + 3.7(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \gamma_3 \frac{0.1(j\omega)^4 + 0.8(j\omega)^3 + 2.7(j\omega)^2 + 3(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} \Bigg| \angle \arctan \Bigg[ 2\gamma_1 \frac{0.6(j\omega)^4 + 3.6(j\omega)^3 + 6.5(j\omega)^2 + 4.5(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + 1.5\gamma_2 \frac{0.1(j\omega)^4 + 1.6(j\omega)^3 + 4.2(j\omega)^2 + 3.7(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \gamma_3 \frac{0.1(j\omega)^4 + 0.8(j\omega)^3 + 2.7(j\omega)^2 + 3(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} \Bigg] \quad (33)$$

As $\omega$ tends to zero, (52) can be approximated as follow:

$$\lim_{\omega \to 0} \tilde{W}(j\omega,\tilde{\nu}) = |2\gamma_1 + 1.5\gamma_2 + \gamma_3| \angle \arctan [2\gamma_1 + 1.5\gamma_2 + \gamma_3] \quad (34)$$

Hence

$$\lim_{\omega \to 0} \tilde{W}(j\omega,\tilde{\nu}) = |2\gamma_1 + 1.5\gamma_2 + \gamma_3| \angle 0^o \quad (35)$$

Applying the *Theorem 3.1*, proposed in Section III, the obtained boundary conditions at low frequencies, are presented in Tab. I. The fuzzy frequency response of the uncertain dynamic system, at low frequencies, presents a range of magnitude in the interval $[0, 6](dB)$ and the phase is $0^o$.

TABLE I
BOUNDARY CONDITIONS AT LOW FREQUENCIES.

| Activated Rule | Boundary Condition | Magnitude (dB) | Phase (Degree) |
|---|---|---|---|
| 1 | $\gamma_1 = 1$; $\gamma_2 = 0$ and $\gamma_3 = 0$ | 6.0206 | $0^o$ |
| 2 | $\gamma_1 = 0$; $\gamma_2 = 1$ and $\gamma_3 = 0$ | 3.5218 | $0^o$ |
| 3 | $\gamma_1 = 0$; $\gamma_2 = 0$ and $\gamma_3 = 1$ | 0 | $0^o$ |

*2) High Frequencies Analysis:* Likewise, from the TS fuzzy dynamic model, Eq. (29), and now applying the concepts seen in the Subsection IV-B, the steady-state response for sinusoidal input at high frequencies for the uncertain dynamic system can be obtained as follow:

$$\lim_{\omega \to \infty} \tilde{W}(j\omega,\tilde{\nu}) = \Bigg| 2\gamma_1 \frac{0.6(j\omega)^4 + 3.6(j\omega)^3 + 6.5(j\omega)^2 + 4.5(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + 1.5\gamma_2 \frac{0.1(j\omega)^4 + 1.6(j\omega)^3 + 4.2(j\omega)^2 + 3.7(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \gamma_3 \frac{0.1(j\omega)^4 + 0.8(j\omega)^3 + 2.7(j\omega)^2 + 3(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} \Bigg| \angle \arctan \Bigg[ 2\gamma_1 \frac{0.6(j\omega)^4 + 3.6(j\omega)^3 + 6.5(j\omega)^2 + 4.5(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + 1.5\gamma_2 \frac{0.1(j\omega)^4 + 1.6(j\omega)^3 + 4.2(j\omega)^2 + 3.7(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \gamma_3 \frac{0.1(j\omega)^4 + 0.8(j\omega)^3 + 2.7(j\omega)^2 + 3(j\omega) + 1}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} \Bigg] \quad (36)$$

In this analysis, the higher degree terms of the transfer functions in the TS fuzzy dynamic model increase more rapidly than the other ones. Thus,

$$\lim_{\omega \to \infty} \tilde{W}(j\omega,\tilde{\nu}) = \left| 2\gamma_1 \frac{0.6(j\omega)^4}{0.1(j\omega)^6} + 1.5\gamma_2 \frac{0.1(j\omega)^4}{0.1(j\omega)^6} + \gamma_3 \frac{0.1(j\omega)^4}{0.1(j\omega)^6} \right| \angle tg^{-1} \left[ 2\gamma_1 \frac{0.6(j\omega)^4}{0.1(j\omega)^6} + 1.5\gamma_2 \frac{0.1(j\omega)^4}{0.1(j\omega)^6} + \gamma_3 \frac{0.1(j\omega)^4}{0.1(j\omega)^6} \right] \quad (37)$$

Hence

$$\lim_{\omega \to \infty} \tilde{W}(j\omega,\tilde{\nu}) = \left| 2\gamma_1 \frac{0.6}{0.1(j\omega)^2} + 1.5\gamma_2 \frac{0.1}{0.1(j\omega)^2} + \gamma_3 \frac{0.1}{0.1(j\omega)^2} \right| \angle -180^o$$

Again applying the *Theorem 3.1*, proposed in Section III, the obtained boundary conditions at high frequencies are presented in Tab. II. The fuzzy frequency response of the uncertain dynamic system, at high frequencies, presents a range of magnitude in the interval $\left[\left|\frac{1}{(j\omega)^2}\right|, \left|\frac{12}{(j\omega)^2}\right|\right] (dB)$ and the phase is $-180^o$.

TABLE II
BOUNDARY CONDITIONS AT HIGH FREQUENCIES.

| Activated Rule | Boundary Condition | Magnitude (dB) | Phase (Degree) |
|---|---|---|---|
| 1 | $\gamma_1 = 1$; $\gamma_2 = 0$ and $\gamma_3 = 0$ | $\left\|12/(j\omega)^2\right\|$ | $-180^o$ |
| 2 | $\gamma_1 = 0$; $\gamma_2 = 1$ and $\gamma_3 = 0$ | $\left\|1.50/(j\omega)^2\right\|$ | $-180^o$ |
| 3 | $\gamma_1 = 0$; $\gamma_2 = 0$ and $\gamma_3 = 1$ | $\left\|0.1/(j\omega)^2\right\|$ | $-180^o$ |

For comparative analysis, the fuzzy frequency response (boundaries conditions at low and high frequencies from Tab. I-II) and frequency response of the uncertain dynamic system are shown in Fig. 5. For this experiment, the frequency response of the uncertain dynamic system was obtained considering the mean of the uncertain parameter $\nu$ in the frequency domain as shown in Fig. 6. It can be seen that the fuzzy frequency response is a region in the frequency domain, defined by the consequent linear sub-models $W^i(s)$, from the operating region of the antecedent space, as demonstrated by the proposed *Theorem 3.1*. This method highlights the efficiency of the fuzzy frequency response to estimate the frequency response of uncertain dynamic systems.

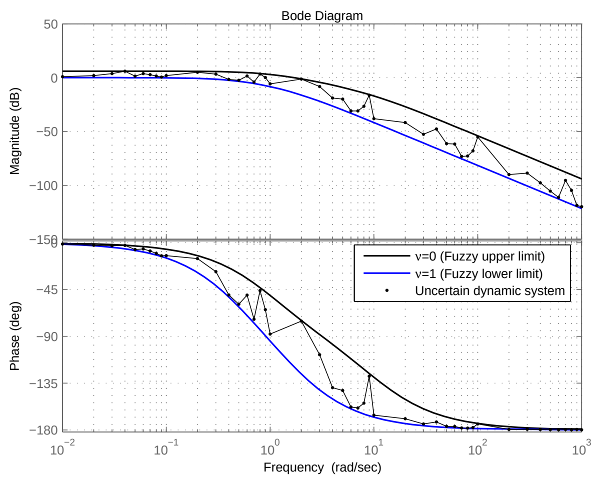


Fig. 5. Comparative analysis between fuzzy frequency response and frequency response of the uncertain dynamic system.

## B. Nonlinear Dynamic System

Now consider the one-link robotic manipulator shown in Fig. 7. The dynamic equation of the one-link robotic manipulator is given by

$$ml^2\ddot{\theta} + d\dot{\theta} + mgl\sin(\theta) = u, \quad (38)$$

with: $m = 1kg$, payload; $l = 1m$, length of the link; $g = 9.81m/s^2$, gravitational constant; $d = 1kgm^2/s$, damping factor and $u$, control variable $(kgm^2/s^2)$.

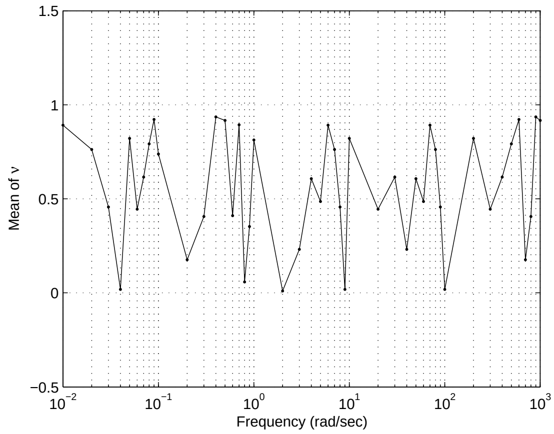


Fig. 6. Mean variation of uncertain parameter $\nu$ in frequency domain.

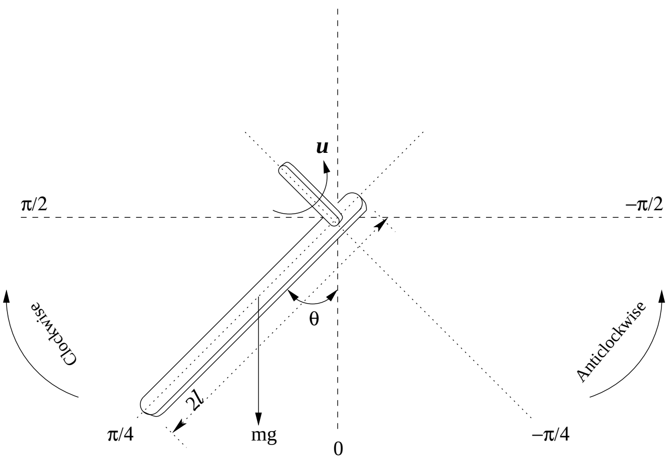


Fig. 7. One-link robotic manipulator.

A LPV model can be obtained from nonlinear model in the Eq. (38) by Taylor series expansion of the nonlinearity $\sin\theta$ in some operating points [10]. For the case that $\nu$ is close to $\nu_0$, it can be able to ignore the higher-order derivative terms. Thus

$$f(\nu) \cong f(\nu_0) + \left.\frac{df(\nu)}{d\nu}\right|_{\nu=\nu_0} (\nu - \nu_0). \quad (39)$$

From Eq. (39), the LPV Plant is

$$ml^2\ddot{\theta} + d\dot{\theta} + mgl\left[a + b\theta\right] = u, \quad (40)$$

where $a = \sin\nu - \nu\cos\nu$; $b = \cos\nu$ and $\nu$ is the scheduling variable that represents the operating point (angle). In terms of transfer function, it has

$$H(s,\nu) = \frac{\Theta(s,\nu)}{U(s,\nu)} = \frac{1}{ml^2s^2 + ds + mgl\cos\nu}, \quad (41)$$

where $U(s,\nu) = U(s) - mgl[\sin\nu - \nu\cos\nu]$. From the LPV model in Eq. (41) and assuming the dynamics of the system in the range of $[-\pi/4, \pi/4]$, it can obtain the TS Fuzzy Model choosing some operating points:

**Sub-model 1 ($\nu = \theta = -\pi/4$):**

$$W^1(s, -\pi/4) = \frac{\Theta(s, -\pi/4)}{U(s, -\pi/4)} = \frac{1}{s^2 + s + 6.9367}. \qquad (42)$$

**Sub-model 2 ($\nu = \theta = 0$):**

$$W^2(s, 0) = \frac{\Theta(s, 0)}{U(s, 0)} = \frac{1}{s^2 + s + 9.81}. \qquad (43)$$

**Sub-model 3 ($\nu = \theta = +\pi/4$):**

$$W^3(s, +\pi/4) = \frac{\Theta(s, +\pi/4)}{U(s, +\pi/4)} = \frac{1}{s^2 + s + 6.9367}. \qquad (44)$$

The TS fuzzy dynamic model rules base results

$$\begin{array}{lllllll} Rule^{(1)}: & \text{IF} & \nu & \text{is} & -\pi/4 & \text{THEN} & W^1(s, -\pi/4) \\ Rule^{(2)}: & \text{IF} & \nu & \text{is} & 0 & \text{THEN} & W^2(s, 0) \\ Rule^{(3)}: & \text{IF} & \nu & \text{is} & +\pi/4 & \text{THEN} & W^3(s, +\pi/4), \end{array} \qquad (45)$$

Again from Eq. (7) the TS fuzzy dynamic model of the one-link robotic manipulator, Eq. (45), is given by

$$\tilde{W}(j\omega, \tilde{\nu}) = \\ = \left| \frac{\gamma_1}{(j\omega)^2 + (j\omega) + 6.9367} + \frac{\gamma_2}{(j\omega)^2 + (j\omega) + 9.81} + \right. \\ \left. + \frac{\gamma_3}{(j\omega)^2 + (j\omega) + 6.9367} \right| \angle \arctan \\ \left\{ \frac{\gamma_1}{(j\omega)^2 + (j\omega) + 6.9367} + \frac{\gamma_2}{(j\omega)^2 + (j\omega) + 9.81} \right. \\ \left. + \frac{\gamma_3}{(j\omega)^2 + (j\omega) + 6.9367} \right\} \qquad (46)$$

$$\tilde{W}(j\omega, \tilde{\nu}) = \\ \left| \frac{[\gamma_1 + \gamma_2 + \gamma_3](j\omega)^2 + [\gamma_1 + \gamma_2 + \gamma_3](j\omega) +}{Den_{[\tilde{W}(j\omega, \tilde{\nu})]}} \right. \\ \left. + \frac{9.81\gamma_1 + 6.9367\gamma_2 + 9.81\gamma_3}{Den_{[\tilde{W}(j\omega, \tilde{\nu})]}} \right| \angle \arctan \\ \left\{ \frac{[\gamma_1 + \gamma_2 + \gamma_3](j\omega)^2 + [\gamma_1 + \gamma_2 + \gamma_3](j\omega) +}{Den_{[\tilde{W}(j\omega, \tilde{\nu})]}} \right. \\ \left. + \frac{9.81\gamma_1 + 6.9367\gamma_2 + 9.81\gamma_3}{Den_{[\tilde{W}(j\omega, \tilde{\nu})]}} \right\} \qquad (47)$$

where

$$Den_{[\tilde{W}(j\omega, \tilde{\nu})]} = (j\omega)^4 + 2(j\omega)^3 + 17.7467(j\omega)^2 + \\ +16.7467(j\omega) + 68.0490 \qquad (48)$$

*1) Low Frequencies Analysis:* From the TS fuzzy dynamic model, Eq. (29), and applying the concepts seen in the subsection IV-A, the steady-state response for sinusoidal input at low frequencies for the one-link robotic manipulator can be obtained as follow:

$$\lim_{\omega \to 0} \tilde{W}(j\omega, \tilde{\nu}) = \\ \left| \frac{[\gamma_1 + \gamma_2 + \gamma_3](j\omega)^2 + [\gamma_1 + \gamma_2 + \gamma_3](j\omega) +}{Den_{[\tilde{W}(j\omega, \tilde{\nu})]}} \right. \\ \left. + \frac{9.81\gamma_1 + 6.9367\gamma_2 + 9.81\gamma_3}{Den_{[\tilde{W}(j\omega, \tilde{\nu})]}} \right| \angle \arctan \\ \left\{ \frac{[\gamma_1 + \gamma_2 + \gamma_3](j\omega)^2 + [\gamma_1 + \gamma_2 + \gamma_3](j\omega) +}{Den_{[\tilde{W}(j\omega, \tilde{\nu})]}} \right. \\ \left. + \frac{9.81\gamma_1 + 6.9367\gamma_2 + 9.81\gamma_3}{Den_{[\tilde{W}(j\omega, \tilde{\nu})]}} \right\} \qquad (49)$$

As $\omega$ tends to zero, Eq. (52) can be approximated as follow:

$$\lim_{\omega \to 0} \tilde{W}(j\omega, \tilde{\nu}) = \left| \frac{9.81\gamma_1 + 6.9367\gamma_2 + 9.81\gamma_3}{68.0490} \right| \angle \arctan \\ \left\{ \frac{9.81\gamma_1 + 6.9367\gamma_2 + 9.81\gamma_3}{68.0490} \right\} \qquad (50)$$

Hence

$$\lim_{\omega \to 0} \tilde{W}(j\omega, \tilde{\nu}) = |0.1442\gamma_1 + 0.1019\gamma_2 + 0.1442\gamma_3| \angle 0^o \qquad (51)$$

Applying the *Theorem 3.1*, proposed in section III, the obtained boundary conditions at low frequencies, are presented in Tab. III. The fuzzy frequency response of the one-link robotic manipulator, at low frequencies, presents a range of magnitude in the interval $[-19.8365; -16.8207](dB)$ and the phase is $0^o$.

TABLE III
BOUNDARY CONDITIONS AT LOW FREQUENCIES

| Activated Rule | Boundary Condition | Magnitude (dB) | Phase (Degree) |
|---|---|---|---|
| 1 | $\gamma_1 = 1$; $\gamma_2 = 0$ and $\gamma_3 = 0$ | $-16.8207$ | $0^o$ |
| 2 | $\gamma_1 = 0$; $\gamma_2 = 1$ and $\gamma_3 = 0$ | $-19.8365$ | $0^o$ |
| 3 | $\gamma_1 = 0$; $\gamma_2 = 0$ and $\gamma_3 = 1$ | $-16.8207$ | $0^o$ |

*2) High Frequencies Analysis:* Likewise, from the TS fuzzy dynamic model, Eq. (29), and now applying the concepts seen in the subsection IV-B, the steady-state response for sinusoidal input at high frequencies for the one-link robotic manipulator can be obtained as follow:

$$\lim_{\omega \to \infty} \tilde{W}(j\omega, \tilde{\nu}) =$$
$$\left| \frac{[\gamma_1 + \gamma_2 + \gamma_3](j\omega)^2 + [\gamma_1 + \gamma_2 + \gamma_3](j\omega) +}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \frac{9.81\gamma_1 + 6.9367\gamma_2 + 9.81\gamma_3}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} \right| \angle \arctan$$
$$\left\{ \frac{[\gamma_1 + \gamma_2 + \gamma_3](j\omega)^2 + [\gamma_1 + \gamma_2 + \gamma_3](j\omega) +}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} + \frac{9.81\gamma_1 + 6.9367\gamma_2 + 9.81\gamma_3}{Den_{[\tilde{W}(j\omega,\tilde{\nu})]}} \right\} \quad (52)$$

In this analysis, the higher degree terms of the transfer functions in the TS fuzzy dynamic model increase more rapidly than the other ones. Thus,

$$\lim_{\omega \to \infty} \tilde{W}(j\omega, \tilde{\nu}) =$$
$$\left| \frac{(\gamma_1 + \gamma_2 + \gamma_3)}{(j\omega)^2} \right| \angle \arctan \left\{ \frac{(\gamma_1 + \gamma_2 + \gamma_3)}{(j\omega)^2} \right\}$$

Hence

$$\lim_{\omega \to \infty} \tilde{W}(j\omega, \tilde{\nu}) = \left| \frac{(\gamma_1 + \gamma_2 + \gamma_3)}{(j\omega)^2} \right| \angle -180^o$$

Again applying the *Theorem 3.1*, proposed in section III, the obtained boundary conditions at high frequencies are presented in Tab. IV. The fuzzy frequency response of the one-link robotic manipulator, at high frequencies, presents the magnitude of $\left| \frac{1}{(j\omega)^2} \right| (dB)$ and the phase is $-180^o$.

TABLE IV

BOUNDARY CONDITIONS AT HIGH FREQUENCIES

| Activated Rule | Boundary Condition | Magnitude (dB) | Phase (Degree) |
|---|---|---|---|
| 1 | $\gamma_1 = 1$; $\gamma_2 = 0$ and $\gamma_3 = 0$ | $\lvert 1/(j\omega)^2 \rvert$ | $-180^o$ |
| 2 | $\gamma_1 = 0$; $\gamma_2 = 1$ and $\gamma_3 = 0$ | $\lvert 1.0/(j\omega)^2 \rvert$ | $-180^o$ |
| 3 | $\gamma_1 = 0$; $\gamma_2 = 0$ and $\gamma_3 = 1$ | $\lvert 1/(j\omega)^2 \rvert$ | $-180^o$ |

For comparative analysis, the fuzzy frequency response (boundaries conditions at low and high frequencies from Tab. I-II) and frequency response of the one-link robotic manipulator are shown in Fig. 8.

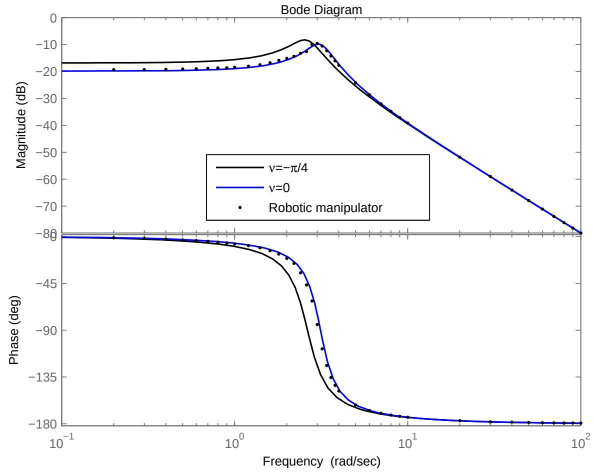


Fig. 8. Comparative analysis between fuzzy frequency response and frequency response of the one-link robotic manipulator.

## VI. Conclusions

The Fuzzy Frequency Response: Definition and Analysis for Complex Dynamic Systems is proposed in this paper. It was shown that the fuzzy frequency response is a region in the frequency domain, defined by the consequent linear sub-models $W^i(s)$, from operating regions of the complex dynamic system, according to the proposed *Theorem 3.1*. This formulation is very efficient and can be used for robust control design for complex dynamic systems.


## Acknowledgment

The authors would like to thank CAPES through the Ph.D. Program UFCG-IFMA. This work was supported in part by the FAPEMA.


## References


[1] Franklin, G. F.; J. D. Powell and Baeini A. E. (1986), "Feedback Control of Dynamic Systems." *Addison-Wesley*, USA.

[2] Phillips, Charles L. and Harbor, Royce D. (1996), "Feedback Control Systems." *Prentice Hall*, 3rd ed., New Jersey.

[3] Ioannou, P. A. and Sun, J. (1996), "Robust Adaptive Control." *Prentice Hall*.

[4] James, H. M.; Nichols, N. B. and Phillips, R. S. (1947), "Theory of servomechanisms." *McGraw-Hill*, Volume 25 of MIT Radiation Laboratory Series, New York.

[5] Nyquist, H. (1932), "Regeneration theory." *Bell Systems Technical Journal*.

[6] Bode, H. W. (1940), "Feedback amplifier design." *Bell Systems Technical Journal*, Vol.19, 1940, pp.42.

[7] A. P. Schust, Jr. (1973), "Determination of Aircraft Response Caracteristics in Approach/Landing Configuration for Microwave Landing System Program," *Report FT-61R-73, Naval Air Test Center*, Patuxent River, MD.

[8] Lascu, C.; Asiminoaei, L.; Boldea, I. and Blaabjerg, F. (2009), "Frequency Response Analysis of Current Controllers for Selective Harmonic Compensation in Active Power Filters," *IEEE Transactions on Industrial Eletronics*, Vol. 56, No. 2.

[9] Takagi, T. and Sugeno, M. (1985), "Fuzzy identification of systems and its applications to modeling and control." *IEEE Trans. Syst. Man. Cyber*, Vol. 15, pp. 116–132.

[10] Serra, G. L. O. and Carlos Cesar T. Ferreira and Joabe A. Silva (2009), "Development Method for a Robust PID Fuzzy Controller of LPV Systems." *IEEE International Conference on Fuzzy Systems*.